

# MARKOV BASES OF LATTICE IDEALS

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**ABSTRACT.** Let  $L \subset \mathbb{Z}^n$  be a lattice,  $R = \mathbb{k}[x_1, \dots, x_n]$  where  $\mathbb{k}$  is a field and  $I_L = \langle x^u - x^v : u - v \in L \rangle$  the corresponding lattice ideal. Most results in the literature on generating sets of lattice ideals concern the case  $L \cap \mathbb{N}^n = \{\mathbf{0}\}$ . In this paper, using appropriate graphs for each fiber, we characterize minimal generating sets of  $I_L$  of minimal cardinality for all lattices and give invariants for these generating sets. As an application we characterize all binomial complete intersection lattice ideals.

## 1. INTRODUCTION

Let  $R = \mathbb{k}[x_1, \dots, x_n]$  where  $\mathbb{k}$  is a field, and let  $L$  be a lattice in  $\mathbb{Z}^n$ . The lattice ideal  $I_L$  is defined to be the ideal generated by the following binomials:

$$I_L := \langle x^u - x^v : u - v \in L \rangle.$$

Let  $\mu(I_L)$  be the least cardinality of any minimal generating set of  $I_L$  consisting of binomials. We call *Markov basis* of  $I_L$  a minimal system of binomial generators of  $I_L$  of cardinality  $\mu(I_L)$ . The study of lattice ideals is a rich subject on its own, see [22, 33] for the general theory and [21] for recent developments. Moreover lattice ideals have applications in diverse areas in mathematics, such as algebraic statistics [8, 26], integer programming [10], hypergeometric differential equations [9], graph theory [25], etc. We note that such ideals were first systematically studied in [11] and that toric ideals are lattice ideals  $I_L$  for which the lattice  $L$  is the kernel of an integer matrix. We note that almost all results in the literature are about lattices  $L$  such that  $L \cap \mathbb{N}^n = \{\mathbf{0}\}$ , with very few exceptions like in [11, 19, 13, 15, 20]. If  $L$  is such that  $L \cap \mathbb{N}^n = \{\mathbf{0}\}$  we say that  $L$  is *positively graded*. Let  $\mathcal{A}$  be the subsemigroup of  $\mathbb{Z}^n/L$  generated by the elements  $\{\mathbf{a}_i = \mathbf{e}_i + L : 1 \leq i \leq n\}$ , where  $\{\mathbf{e}_i : 1 \leq i \leq n\}$  is the canonical basis of  $\mathbb{Z}^n$  and set

$$\deg_{\mathcal{A}}(x^v) := v_1 \mathbf{a}_1 + \dots + v_n \mathbf{a}_n \in \mathcal{A}$$

where  $x^v = x_1^{v_1} \dots x_n^{v_n}$ . It follows that

$$I_L = \langle x^u - x^v : \deg_{\mathcal{A}}(x^u) = \deg_{\mathcal{A}}(x^v) \rangle$$

and that  $I_L$  is  $\mathcal{A}$ -graded. When  $L$  is positively graded, the semigroup  $\mathcal{A}$  is partially ordered:

$$\mathbf{c} \geq \mathbf{d} \iff \text{there is } \mathbf{e} \in \mathcal{A} \text{ such that } \mathbf{c} = \mathbf{d} + \mathbf{e}.$$

Then the grading of  $\mathcal{A}$  forces the  $I_L$ -fiber of  $x^u$ , i.e. the set  $\{x^v : x^v - x^u \in I_L\} = \{x^v : \deg_{\mathcal{A}}(x^v) = \deg_{\mathcal{A}}(x^u)\}$ , to be finite. The homogeneous Nakayama Lemma applies and guarantees that all minimal binomial generating sets of  $I_L$  are Markov bases of  $I_L$ , since they have the same cardinality. Let  $S$  be a Markov basis of  $I_L$

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and form the multiset of all  $I_L$ -fibers corresponding to the elements of  $S$ . This multiset is an invariant of  $I_L$  and does not depend on the choice of  $S$ . Moreover the binomials of  $S$  are primitive, see [33, 22], and thus  $S$  is a subset of the Graver basis of  $I_L$ . Since the Graver basis of  $I_L$  is a finite set, see [22], it follows that the Universal Markov basis of  $I_L$ , (see [17]), i.e. the union of all Markov bases, is finite.

The situation for a general lattice ideal is completely different. Take for example the lattice  $L$  generated by  $\{(1, 1), (5, 0)\}$ . It can be shown that the following are minimal generating sets of  $I_L$  in  $\mathbb{k}[x, y]$ :  $\{1 - xy, 1 - x^5\}$ ,  $\{1 - xy, x^3 - y^2\}$ ,  $\{1 - x^2y^2, 1 - x^3y^3, 1 - x^5\}$ . It is clear that  $I_L$  is not a principal ideal and that  $\mu(I_L) = 2$ . It is not hard to produce minimal generating sets of  $I_L$  of any desired cardinality greater than 2. For example let  $p_1, \dots, p_s$  be  $s$  distinct primes and let  $a_i = p_1 \cdots p_s / p_i$ . The elements  $a_1, \dots, a_s$  are relatively prime and the greatest common divisor of  $(1 - z^{a_1}, \dots, 1 - z^{a_s})$  is  $1 - z$ , while the greatest common divisor of  $\{1 - z^{a_j} : j \neq i\}$  is  $1 - z^{p_i}$ . It follows that  $\langle 1 - (xy)^{a_1}, \dots, 1 - (xy)^{a_s} \rangle = \langle 1 - xy \rangle$  and that  $\{1 - x^5, 1 - (xy)^{a_1}, \dots, 1 - (xy)^{a_n}\}$  is a minimal generating set of  $I_L$ . Even if we restrict our attention to Markov bases of  $I_L$ , we get some very interesting behaviour. The set  $\{1 - x^{2012}y^{2017}, y^4 - x^{2013}y^{2022}\}$  is a Markov basis of  $I_L$ , but it is easily seen that its elements are not primitive binomials. It is easy to produce an infinite set of Markov bases of  $I_L$ : in Section 4 we discuss how to obtain Markov bases of lattice ideals in general. It follows that in this example, the Universal Markov basis of  $I_L$  is infinite. Moreover, there is no unique multiset of  $I_L$ -fibers corresponding to the Markov bases of  $I_L$ . Indeed the monomials of  $\mathbb{k}[x, y]$  are partitioned into exactly five infinite  $I_L$ -fibers:

$$F_k = \{x^i y^j : i - j \equiv k \pmod{5}, \quad 0 \leq k \leq 4\}.$$

The multiset of the  $I_L$ -fibers for  $\{1 - xy, 1 - x^5\}$  is  $\{F_0, F_0\}$  while the multiset of the  $I_L$ -fibers for  $\{1 - xy, x^3 - y^2\}$  is  $\{F_0, F_3\}$ .

Algorithms for computing a generating set for lattice ideals were given in [16, 2, 15]. The main problem we address in this paper is how to determine invariants of Markov bases of a lattice ideal  $I_L$  and how to detect whether a set of binomials of  $I_L$  is a Markov basis of  $I_L$ . Instead of considering the isolated fibers of  $I_L$  we consider equivalence classes of fibers and show that for all Markov bases of  $I_L$ , the multiset of equivalence classes is an invariant of  $I_L$ .

In the last years, due to applications of Markov bases to Algebraic Statistics, there is an interest in determining the *indispensable binomials* of a lattice ideal, see [25, 26, 4, 1, 27]. Of particular interest is the case when all elements in the Universal Markov basis of the lattice ideal are indispensable as is the case for generic lattice ideals, see [28]. An indispensable binomial is a binomial that appears in every Markov basis of the lattice ideal up to a constant multiple. When the lattice is positively graded the problem of determining such binomials has been completely solved, see [4]. In this paper we address this problem for general lattice ideals. We show that if the lattice  $L$  is *not* positively graded, then there is at most one indispensable binomial.

Another question we address is characterizing binomial complete intersection lattice ideals. We recall that a lattice ideal  $I_L$  of height  $r$  is a *complete intersection* if there exist polynomials  $P_1, \dots, P_r$  such that  $I_L = \langle P_1, \dots, P_r \rangle$  and  $I_L$  is a *binomial complete intersection* if there exist binomials  $B_1, \dots, B_r$  such that  $I_L = \langle B_1, \dots, B_r \rangle$ . If Nakayama's lemma applies then complete intersection lattice ideals are automatically binomial complete intersections. The problem is completely

solved when  $L$  is positively graded by a series of articles: [14, 7, 32, 18, 36, 24, 30, 29, 12, 31, 23]. The final conclusion is that  $I_L$  is a complete intersection if and only if the matrix  $M$  whose rows correspond to a basis of  $L$  is “mixed dominating”: every row of  $M$  has a positive and negative entry and  $M$  contains no square submatrix with this property. When  $L$  is not positively graded the situation is less clear. In this paper we characterize all binomial complete intersection lattice ideals.

The structure of this paper is as follows: in Section 2 given the lattice  $L \subset \mathbb{Z}^n$  we introduce and examine the properties of  $L_{\text{pure}}$ , a sublattice of  $L$  which will be crucial in our study. Of particular importance is the support of  $L_{\text{pure}}$  which we denote by  $\sigma_L$ . In Section 3 we define an equivalence relation among the  $I_L$ -fibers. We order the resulting equivalence classes and show that any descending chain of such classes stabilizes. We prove that the multiset of *equivalence classes of fibers* corresponding to a Markov basis of  $I_L$  is an invariant of  $I_L$ . In Section 4 we characterize all minimal binomial generating sets of pure lattices. Then we describe all Markov bases of  $I_L$  for any lattice  $L$ . We explicitly compute  $\mu(I_L)$  in terms of a related graph. We show that if  $\text{rank } L_{\text{pure}} \geq 1$ , there is at most one indispensable binomial. In Section 5 we characterize all lattices  $L$  such that  $I_L$  is a binomial complete intersection. In section 6 we work out an example in full detail.

## 2. FIBERS, THE PURE SUBLATTICE AND BASES OF A LATTICE

Let  $R = \mathbb{k}[x_1, \dots, x_n]$  where  $\mathbb{k}$  is a field,  $L$  a lattice in  $\mathbb{Z}^n$ ,  $I_L = \langle x^u - x^v : u - v \in L \rangle$ . We denote by  $\mathbb{T}^n$  the set of monomials of  $R$  including  $1 = x^0$ . If  $J$  is a monomial ideal of  $R$  we denote by  $G(J)$  the unique minimal set of monomial generators of  $J$ . For  $r \in \mathbb{N}$  we let  $[r] = \{1, \dots, r\}$ . Let  $a = (a_1, \dots, a_n)$ ,  $b = (b_1, \dots, b_n) \in \mathbb{Z}^n$ . We write  $a \geq b$  if  $a_i \geq b_i$  for  $i = 1, \dots, n$ . We write  $a \geq \mathbf{0}$  if  $a \in \mathbb{N}^n$ . If either  $a \geq \mathbf{0}$  or  $-a \geq \mathbf{0}$  we say that  $a$  is *pure*. We say that  $a, b$  are incomparable if  $a - b$  is not pure. In general we let  $\text{supp}(a) = \{i : a_i \neq 0\} \subset [n]$ . For any subset  $X$  of  $\mathbb{Z}^n$  we let

$$\text{supp}(X) := \bigcup_{w \in X} \text{supp}(w) .$$

**Definition 2.1.** We say that  $F \subset \mathbb{T}^n$  is an  $I_L$ -*fiber* if there exists  $x^u \in \mathbb{T}^n$  such that  $F = \{x^v \in \mathbb{T}^n : v - u \in L\}$ . If  $x^u \in F$ , and  $F$  is an  $I_L$ -fiber we write  $F_u$  or  $F_{x^u}$  for  $F$ . If  $B \in I_L$  and  $B = x^u - x^v$  we write  $F_B$  for  $F_u$ . When  $F$  is an  $I_L$ -fiber we let  $M_F = \langle x^u : x^u \in F \rangle$  be the monomial ideal generated by the elements of  $F$ .

From the properties of the lattice and the definition of lattice ideals we get the following:

**Proposition 2.2.** *If  $x^v \in F_u$  then  $F_u = F_v$ . Moreover  $F_u = \{x^v : x^v - x^u \in I_L\}$ . If  $x^u - x^v \in I_L$  then  $u - v \in L$ .*

We remark that  $F_u$  is a singleton if and only if there is no binomial  $0 \neq B \in I_L$  such that  $F_B = F_u$ . We note that  $F \subset M_F$  and  $G(M_F) \subset F$ . The following proposition follows also from [22, Theorem 8.6].

**Proposition 2.3.** *Let  $L \subset \mathbb{Z}^n$  be a lattice. The following are equivalent:*

- (1) *The lattice  $L$  contains a nonzero pure element.*
- (2) *All  $I_L$ -fibers are infinite.*
- (3) *There exists an  $I_L$ -fiber which is infinite.*

*Proof.* (1)  $\Rightarrow$  (2) Let  $F$  be an  $I_L$ -fiber and suppose that  $\mathbf{0} \neq u \in L \cap \mathbb{N}^n$ . It is easy to see that if  $v \in \mathbb{N}^n$  and  $F$  is the  $I_L$ -fiber such that  $x^v \in F$ , then  $x^{v+lu} \in F$  for all  $l \in \mathbb{N}$ . Thus  $F$  is infinite. (2)  $\Rightarrow$  (3) obvious. (3)  $\Rightarrow$  (1) Suppose that an  $I_L$ -fiber  $F$  is infinite. Let  $x^v \in F$  be such that  $x^v \notin G(M_F)$ . Note that since  $F$  is infinite such a  $v$  exists. Since  $x^v \in M_F$ , there exists a monomial  $x^u \in G(M_F)$  such that  $x^u | x^v$  and thus  $x^v = x^w x^u$  for  $\mathbf{0} \neq w \in \mathbb{N}^n$ . Since  $x^v, x^u \in F$ , it follows that  $w = v - u \in L$ , therefore  $w \in L \cap \mathbb{N}^n$ .  $\square$

**Corollary 2.4.** *Let  $L \subset \mathbb{Z}^n$  be a lattice. The lattice  $L$  is positively graded (i.e.  $L \cap \mathbb{N}^n = \{\mathbf{0}\}$ ) if and only if  $G(M_F) = F$  where  $F$  is any  $I_L$ -fiber.*

*Proof.* Suppose that  $L \cap \mathbb{N}^n = \{\mathbf{0}\}$ . Since  $M_F = \langle F \rangle$  to prove that  $G(M_F) = F$ , it is enough to show that if  $x^a \neq x^b \in F$  then  $a, b$  are incomparable. Suppose otherwise. Then  $a - b \in L$  is pure, a contradiction. For the other direction suppose that  $G(M_F) = F$ . Thus  $F$  is finite and the conclusion follows from Proposition 2.3.  $\square$

**Notation 2.5.** We let  $L^+ = L \cap \mathbb{N}^n$ ,  $\sigma_L = \text{supp}(L^+)$  and  $L_{\text{pure}}$  be the subgroup of  $L$  generated by  $L^+$ .

In the course of the proof of Proposition 2.3 we proved the following:

**Proposition 2.6.** *Let  $L \subset \mathbb{Z}^n$  be a lattice and let  $F$  be an  $I_L$ -fiber. If  $G(M_F) = \{x^{a_1}, \dots, x^{a_s}\}$  then*

$$F = \bigcup_{i=1}^s \{x^{a_i} x^w : w \in L^+\}.$$

The next proposition considers the support of the elements of  $L$  that belong to  $L_{\text{pure}}$ .

**Proposition 2.7.** *There exists an element  $w$  in  $L^+$  such that  $\text{supp}(w) = \sigma_L$ . For  $u \in L$  we have that  $\text{supp}(u) \subset \sigma_L$  if and only if  $u \in L_{\text{pure}}$ .*

*Proof.* The existence of  $w$  follows from the observation that if  $w_1, w_2 \in L^+$  then  $w_1 + w_2 \in L^+$  and  $\text{supp}(w_1) \cup \text{supp}(w_2) = \text{supp}(w_1 + w_2)$ .

Suppose now that  $u \in L$  and  $\text{supp}(u) \subset \sigma_L$ . Let  $w \in L^+$  be such that  $\text{supp}(w) = \sigma_L$ . It is clear that for  $l \in \mathbb{N}$ ,  $l \gg 0$ ,  $u + lw = w' \in \mathbb{N}^n$ . Since  $u, lw \in L$  it follows that  $w' \in L$  and thus  $w' \in L^+$ . Therefore  $u = w' - lw \in L_{\text{pure}}$ .  $\square$

Since  $L_{\text{pure}}$  is generated by the elements of  $L^+$  it is clear that

$$\text{supp}(L_{\text{pure}}) = \sigma_L.$$

Let  $u = (u_i) \in \mathbb{Z}^n$ . By  $u^{\sigma_L}$  we mean the vector  $(u_i)_{i \notin \sigma_L}$ . The following is an immediate consequence of Proposition 2.7.

**Corollary 2.8.** *Let  $L \subset \mathbb{Z}^n$  be a lattice and  $u \in L$ . Then  $u \in L_{\text{pure}}$  if and only if  $u^{\sigma_L} = \mathbf{0}$ .*

**Definition 2.9.** A nonzero vector  $u \in L$  is called  $L$ -primitive if whenever  $\lambda u \in L$  where  $\lambda \in \mathbb{Q}$  then  $\lambda \in \mathbb{Z}$ . In other words,  $u$  is  $L$ -primitive if and only if  $\mathbb{Q}u \cap L = \mathbb{Z}u$ .

Equivalently  $u$  is  $L$ -primitive if it is the “smallest” element of  $L$  in the direction determined by  $u$ .

**Proposition 2.10.** *Let  $\mathbf{0} \neq v \in L$ . There is an  $L$ -primitive vector  $u \in L$  such that  $v = \lambda u$  for  $\lambda \in \mathbb{Z}$ .*

*Proof.* If  $v$  is not  $L$ -primitive there is  $v' \in L$  such that  $v' = \frac{k}{m}v$  where  $m \neq 1$  and  $\gcd(k, m) = 1$ . Thus  $mv' = kv$  and  $m$  divides all coordinates of  $v$ . Moreover there are  $t_1, t_2 \in \mathbb{Z}$  such that  $1 = t_1k + t_2m$ . It follows that

$$\frac{1}{m}v = t_1v' + t_2v \implies u = \frac{1}{m}v \in L.$$

We note that  $v$  is an integer multiple of  $u$ . If  $u$  is not  $L$ -primitive we repeat this procedure. Since  $v \in \mathbb{Z}^n$  this procedure has to end in a finite number of steps.  $\square$

Consider now any basis of  $L$  as a  $\mathbb{Z}$ -module. The next theorem states that the elements of such a basis are necessarily  $L$ -primitive.

**Theorem 2.11.** *Let  $L \subset \mathbb{Z}^n$  be a lattice and let  $\mathbb{B}$  be a basis of  $L$  as a  $\mathbb{Z}$ -module. The elements of  $\mathbb{B}$  are  $L$ -primitive.*

*Proof.* Since  $L$  is a sublattice of  $\mathbb{Z}^n$ , there exists an  $r \in \mathbb{N}$  such that  $L \cong \mathbb{Z}^r$ ,  $r = \text{rank}(L)$ . Let  $\mathbb{B} = \{u_1, \dots, u_r\}$ . Suppose that for some  $i \in [r]$ ,  $u_i$  is not  $L$ -primitive. By Proposition 2.10 it follows that there is  $v \in L$  and  $m \in \mathbb{N}$ ,  $m \neq 1$ , such that  $v = \frac{1}{m}u_i$ . Since  $\mathbb{B}$  is a basis of  $L$  it follows that  $v = \sum \lambda_j u_j$  where  $\lambda_j \in \mathbb{Z}$  for  $j \in [r]$ . Since the  $\mathbb{Q}$ -coordinates of  $v$  are unique it follows that  $\lambda_i = \frac{1}{m}$  and  $\lambda_j = 0$  for  $j \in [r] \setminus \{i\}$  and thus  $\lambda_i = m = 1$ , a contradiction.  $\square$

We consider the usual Euclidean inner product in  $\mathbb{Z}^n$ : if  $a = (a_i)$ ,  $b = (b_i)$  we let  $a \cdot b = \sum a_i b_i$ .

**Theorem 2.12.** *Let  $L$  be a lattice and  $u_1$  an  $L$ -primitive vector. There exists a basis  $\mathbb{B}$  of  $L$  such that  $u_1 \in \mathbb{B}$ .*

*Proof.* We will do induction on  $r$ , the rank of  $L$ . If  $r = 1$  we are done. Assume  $r > 1$ . Let  $w_1$  be a vector in  $\mathbb{Z}^n$  such that  $w_1 \cdot u_1 = 0$ . Since  $r > 1$  it is clear that there is  $u_2 \in L$  such that  $w_1 \cdot u_2 \in \mathbb{N}$ . Choose  $u_2 \in L$  to be such that  $w_1 \cdot u_2$  is positive and as small as possible. We will show that for  $u \in L$  there is a  $\lambda \in \mathbb{Z}$  such that  $w_1 \cdot u = \lambda(w_1 \cdot u_2)$ . Suppose not. Then

$$w_1 \cdot u = q(w_1 \cdot u_2) + r, \quad 0 \leq r \leq w_1 \cdot u_2.$$

It follows that  $w_1 \cdot (u - qu_2) = r$  which contradicts the choice of  $u_2$ . Notice also that  $u_1, u_2$  are linearly independent. We continue this way and obtain a sequence of linearly independent vectors  $u_1, \dots, u_r$  and a sequence of vectors  $w_1, \dots, w_{r-1}$  that satisfy the following properties for  $1 \leq i \leq r-1$ :

- a)  $w_i \cdot u_j = 0$  for  $j = 1, \dots, i$
- b)  $w_i \cdot u_{i+1} > 0$  and
- c) if  $u \in L$  then there is a  $\lambda \in \mathbb{Z}$  such that  $w_i \cdot u = \lambda(w_i \cdot u_{i+1})$ .

It is clear that  $\{u_1, \dots, u_r\}$  is a  $\mathbb{Q}$ -basis of  $L$ . We show that  $\{u_1, \dots, u_r\}$  is a  $\mathbb{Z}$ -basis of  $L$ . Let  $u \in L$ ,  $u = \sum_{i=1}^r \lambda_i u_i$  where  $\lambda_i \in \mathbb{Q}$ . Consider  $w_{r-1} \cdot u$ . Then

$$w_{r-1} \cdot u = \lambda_r(w_{r-1} \cdot u_r).$$

By c) above it follows that  $\lambda_r \in \mathbb{Z}$ . Next consider  $u' = u - \lambda_r u_r = \sum_{i=1}^{r-1} \lambda_i u_i$ . Since  $w_{r-1} \cdot u' = \lambda_{r-1}(w_{r-1} \cdot u_{r-1})$  it follows as above that  $\lambda_{r-1} \in \mathbb{Z}$ . In this way we get that  $\lambda_r, \dots, \lambda_2 \in \mathbb{Z}$ . Consider now  $v = u - \sum_{i=2}^r \lambda_i u_i = \lambda_1 u_1$ . Since  $v \in L$  and  $u_1$  is  $L$ -primitive it follows that  $\lambda_1 \in \mathbb{Z}$ .  $\square$

**Corollary 2.13.** *Let  $L$  be a lattice. There exists a basis of  $L_{\text{pure}}$  whose elements are in  $L^+$  and have support equal to  $\sigma_L$ .*

*Proof.* By Proposition 2.7 and Proposition 2.10 there is an  $L$ -primitive vector  $u_1 \in L^+$  such that  $\text{supp}(u_1) = \sigma_L$ . By Theorem 2.12 there exists a basis  $\{u_1, \dots, u_r\}$  of  $L_{\text{pure}}$ . It is clear that for  $l \gg 0$ ,  $u'_i = u_i + lu_1 \in L^+$  for  $i = 2, \dots, r$ . The set  $\{u_1, u'_2, \dots, u'_r\}$  has the desired properties.  $\square$

Bases of the lattice  $L$  are clearly important for the study of  $I_L$ . We note though that it is well known that it is not enough to compute a basis for  $L$  to find a generating set of  $I_L$ . Indeed we can associate to each  $u \in L$  the polynomial  $B_u = x^{u^+} - x^{u^-}$  where  $u = u^+ - u^-$  and  $u^+, u^- \in \mathbb{N}^n$ . Of course there are many binomials  $x^{w_1} - x^{w_2}$  such that  $u = w_1 - w_2$  and they all belong to  $I_L$ , but  $B_u$  and  $-B_u$  are the only ones with the property that its monomial terms are relatively prime: in some sense  $B_u$  is the lowest term binomial that corresponds to  $u$ . It is a basic fact that  $I_L = \langle B_u : u \in L \rangle$ . Let  $E$  be a basis of  $L$  and define  $I(E) = \langle B_u : u \in E \rangle$ . Clearly  $I(E) \subset I_L$  but the equality does not have to hold. For example let  $L$  be the lattice of  $\mathbb{Z}^4$  with basis  $E = \{u_1 = (1, -1, -1, 1), u_2 = (1, -2, 2, -1)\}$  and let  $B_1 = xw - yz$ ,  $B_2 = xz^2 - y^2w$ . The lattice  $L$  corresponds to the Macaulay curve:

$$L = \ker \begin{bmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$

We see that  $u = (2, -3, 1, 0) \in L$  since  $u = u_1 + u_2$  and that the polynomial  $B = x^2z - y^3$  of  $\mathbb{k}[x, y, z, w]$  is in  $I_L$ . However  $B$  does not belong to  $I(E) = \langle B_1, B_2 \rangle$  since there is no way to create  $x^2z$  from the monomial terms of  $B_1$  and  $B_2$ . The problem is that the relation

$$(2, -3, 1, 0) = (1, -1, -1, 1) + (1, -2, 2, -1)$$

does not *translate* to a relation among the *lowest term* binomials. However the above relation on the elements of  $L$  *translates* to the following relation on elements of  $I(E)$ :

$$wB = xzB_1 + yB_2.$$

Indeed, as it is shown in [33, Lemma 12.2], for toric ideals the following relation holds:

$$I_L = I(E) : (x_1 \cdots x_n)^\infty,$$

where for  $f \in \mathbb{k}[x_1, \dots, x_n]$  and  $J$  an ideal of  $\mathbb{k}[x_1, \dots, x_n]$ ,

$$J : f^\infty = \{g \in \mathbb{k}[x_1, \dots, x_n] : gf^r \in J, r \in \mathbb{N}\}.$$

In the next two sections we are going to describe minimal generating sets of lattice ideals using properties of the  $I_L$ -fibers and of the bases of  $L_{\text{pure}}$ .

### 3. FIBERS AND MARKOV BASES OF LATTICE IDEALS

Let  $R = \mathbb{k}[x_1, \dots, x_n]$  where  $\mathbb{k}$  is a field,  $L \subset \mathbb{Z}^n$  a lattice. For simplicity of notation from now on we write

$$I := I_L, \quad \sigma := \sigma_L.$$

If  $G \subset \mathbb{T}^n$  and  $t \in \mathbb{N}^n$  we let  $x^t G := \{x^t x^u : x^u \in G\}$ .

**Lemma 3.1.** *Let  $G, F$  be  $I$ -fibers. If there exists  $w_1 \in \mathbb{N}^n$ ,  $x^{w_1} \in G$  such that  $x^{w_1} x^u \in F$  then  $x^{w_1} G \subset F$ . Moreover if  $x^{w_2} F \subset G$  for  $w_2 \in \mathbb{N}^n$  then  $w_1 + w_2 \in L^+$  and  $\text{supp}(w_i) \subset \sigma$ ,  $i = 1, 2$ .*

*Proof.* Suppose that  $x^{w_1}x^u = x^v \in F$  and let  $x^{u'} \in G$ . Since  $u' - u \in L$  it follows that  $(w_1 + u') - v \in L$  and thus  $x^{w_1+u'} \in F$ .

If in addition  $x^{w_2}F \subset G$  it follows that  $x^{w_1+w_2}F \subset F$ . Since  $x^{w_1+w_2}x^v, x^v \in F$  it follows that  $w_1 + w_2 \in L \cap \mathbb{N}^n$ .  $\square$

**Definition 3.2.** Let  $F, G$  be  $I$ -fibers. We say that  $F \equiv_I G$  if there exist  $u, v \in \mathbb{N}^n$  such that  $x^u F \subset G$  and  $x^v G \subset F$ .

It is immediate that  $F \equiv_I G$  is an equivalence relation among the  $I$ -fibers. We denote the equivalence class of  $F$  by  $\overline{F}$ . Thus

$$\overline{F} = \{G : G \text{ is an } I\text{-fiber, } G \equiv_I F\}.$$

We note that  $F \equiv_I G$  implies that the cardinality of  $F$  is equal to the cardinality of  $G$ .

**Lemma 3.3.** If  $L_{\text{pure}} = \{\mathbf{0}\}$  and  $F$  is an  $I$ -fiber then  $\overline{F} = \{F\}$ .

*Proof.* By Proposition 2.3,  $|F| < \infty$ . Let  $G$  be an  $I$ -fiber,  $G \equiv_I F$ . There are  $u, v \in \mathbb{N}^n$  such that  $x^u F \subset G$  and  $x^v G \subset F$ . Since  $|F| = |x^u F| = |G| = |x^v G|$  it follows that  $x^v x^u F = F$  and  $x^v = x^u = 1$ .  $\square$

Next we want to investigate the number of equivalent fibers inside each equivalence class when  $L_{\text{pure}} \neq \{\mathbf{0}\}$  and  $\sigma \neq \emptyset$ . If  $u \in \mathbb{Z}^n$  we let  $u_\sigma = (u_i)_{i \in \sigma}$ . If  $s = |\sigma|$  we can assume that  $u_\sigma \in \mathbb{Z}^s$  and then consider the sublattice  $(L_{\text{pure}})_\sigma$  of  $\mathbb{Z}^s$  generated by the vectors  $u_\sigma, u \in L_{\text{pure}}$ . First we note the following:

**Remark 3.4.** Let  $\sigma \neq \emptyset$ ,  $F$  an  $I$ -fiber and  $u \in \mathbb{N}^n$  such that  $\text{supp}(u) \subset \sigma$ . If  $G$  is an  $I$ -fiber with the property  $x^u F \subset G$  then  $G \in \overline{F}$ .

*Proof.* Let  $w \in L^+$  be such that  $\text{supp}(w) = \sigma$ . There exists  $l \gg 0$  such that  $lw - u \in \mathbb{N}^n$ . Since  $lw \in L$  it follows that  $x^{lw} F \subset F$ . Let  $x^v \in G$  such that  $x^v = x^u x^p$  for  $x^p \in F$ . It follows that  $x^{lw-u} x^v = x^{lw} x^p \in F$  and thus  $x^{lw-u} G \subset F$  by Lemma 3.1.  $\square$

**Proposition 3.5.** Let  $L_{\text{pure}} \neq \{\mathbf{0}\}$  and  $F$  an  $I$ -fiber. The cardinality of  $\overline{F}$  is equal to  $|\mathbb{Z}^s / (L_{\text{pure}})_\sigma|$ , where  $s = |\sigma|$ .

*Proof.* For every  $G \in \overline{F}$  choose  $u_G \in \mathbb{N}^n$  such that  $x^{u_G} F \subset G$ . Let

$$\phi : \overline{F} \rightarrow \mathbb{Z}^s / (L_{\text{pure}})_\sigma, \quad \phi(G) = (u_G)_\sigma + (L_{\text{pure}})_\sigma.$$

The definition of  $\phi$  is independent of the choice of  $u_G$ . Indeed suppose that  $u, v \in \mathbb{N}^n$  are such that  $x^u F \subset G$  and  $x^v F \subset G$ . This implies that  $u - v \in L$ . By Lemma 3.1 it follows that  $u^\sigma = v^\sigma = \mathbf{0}$  and by Proposition 2.7 it follows that  $u - v \in L_{\text{pure}}$  and  $u_\sigma - v_\sigma \in (L_{\text{pure}})_\sigma$ .

We will show that  $\phi$  is a bijection: the only part needing proof is the surjection of  $\phi$ . Let  $u' + (L_{\text{pure}})_\sigma$  be an element of  $\mathbb{Z}^s + (L_{\text{pure}})_\sigma$ . First we remark that we can assume without loss of generality that  $u' \in \mathbb{N}^s$ . Indeed, let  $w \in L^+$  be such that  $|\text{supp}(w_\sigma)| = s$ . It is clear that for  $l \gg 0$ ,  $lw_\sigma + u' \in \mathbb{N}^s$  and thus  $u' + (L_{\text{pure}})_\sigma = (lw + u') + (L_{\text{pure}})_\sigma$ . Let  $u \in \mathbb{N}^n$  be such that  $u^\sigma = \mathbf{0}$ ,  $u_\sigma = u'$  and let  $G$  be the  $I$ -fiber such that  $x^u F \subset G$ . By Remark 3.4 it follows that  $G \in \overline{F}$ , and thus  $\phi(G) = u' + (L_{\text{pure}})_\sigma$ .  $\square$

**Examples 3.6.**

- (a) We consider the lattice ideal  $I = \langle 1 - xy \rangle \subset \mathbb{k}[x, y]$  where  $L = \langle (1, 1) \rangle \subset \mathbb{Z}^2$ . There are infinitely many  $I$ -fibers: for any  $c \in \mathbb{Z}$  the set  $F_c = \{x^i y^j : i - j = c\}$  is an  $I$ -fiber. All  $I$ -fibers are infinite and belong to the same equivalence class: the cardinality of this equivalence class is  $|\mathbb{Z}|$ . Indeed  $L_{\text{pure}} = L$ , and  $\mathbb{Z}^2 / L \cong \mathbb{Z}$ .
- (b) If we consider the lattice ideal  $I = \langle 1 - xy, 1 - x^5 \rangle \subset \mathbb{k}[x, y]$ , where  $L = \langle (1, 1), (5, 0) \rangle \subset \mathbb{Z}^2$  then there are exactly five infinite  $I$ -fibers:

$$F_k = \{x^i y^j : i - j \equiv k \pmod{5}, \quad 0 \leq k \leq 4\},$$

which are all equivalent. Hence we have only one equivalence class  $\overline{F_0} = \{F_0, \dots, F_4\}$  which has five equivalent fibers. Indeed  $L_{\text{pure}} = L$  and  $\mathbb{Z}^2 / L \cong \mathbb{Z}_5$ .

We define the relation “ $\leq_I$ ” among the equivalence classes of  $I$ -fibers.

**Definition 3.7.** Let  $F, G$  be  $I$ -fibers. We say that  $\overline{F} \leq_I \overline{G}$  if there exists  $u \in \mathbb{N}^n$  such that  $x^u F \subset G$ .

It is immediate that “ $\leq_I$ ” is well defined and is a partial order among the equivalence classes of  $I$ -fibers. For simplicity of notation we occasionally write  $F \leq_I G$  if  $\overline{F} \leq_I \overline{G}$  and  $F <_I G$  if  $\overline{F} \leq_I \overline{G}$  and  $\overline{F} \neq \overline{G}$ . We note that  $F_{\{1\}} \leq_I F$  for any  $I$ -fiber  $F$ . We also remark that if  $L_{\text{pure}} = \{\mathbf{0}\}$  then  $<_I$  gives the ordering on the fibers of  $I$  induced by the  $\mathbb{Z}^n / L$ -degrees, see [4].

**Theorem 3.8.** Any descending chain of equivalence classes of  $I$ -fibers is finite.

*Proof.* Let

$$\overline{F_1} >_I \dots >_I \overline{F_k} >_I \overline{F_{k+1}} >_I \dots$$

a chain of equivalence classes of fibers with no least element. Choose a representative  $F_i$ ,  $i \in \mathbb{N}$  for each class. Next consider the corresponding ascending chain of monomial ideals:

$$M_{F_1} \subset \dots \subset M_{F_1} + \dots + M_{F_k} \subset M_{F_1} + \dots + M_{F_{k+1}} \subset \dots$$

The chain stabilizes at some step, say  $s$ , so that

$$M_{F_1} + \dots + M_{F_s} = M_{F_1} + \dots + M_{F_{s+1}}.$$

Let  $x^a \in G(M_{F_{s+1}})$ . By the above equality it follows that  $x^a \in M_{F_i}$  for some  $1 \leq i < s+1$  and  $x^a = x^u x^b$  where  $x^b \in G(M_{F_i})$ . Since  $x^u x^b \in F_{s+1}$  it follows that  $x^u F_i \subset F_{s+1}$ . This leads to a contradiction since  $\overline{F_{s+1}} <_I \overline{F_i}$ .  $\square$

**Definition 3.9.** A minimal generating set of  $I$  of minimal cardinality is called a *Markov basis* of  $I$ . Let  $F$  be an  $I$ -fiber. We say that  $F$  is a *Markov  $I$ -fiber* if there exists a Markov basis  $S$  for  $I$  such that  $\overline{F} = \overline{F_B}$  for some  $B$  in  $S$ .

If  $L_{\text{pure}} = \{\mathbf{0}\}$  then  $I$  is a positively graded lattice ideal. This is the case dealt in [4] and [10]. In order to specify a Markov basis of  $I$ , certain subideals of  $I$  were considered, one for each  $I$ -fiber  $F$ . We generalize these constructions for arbitrary lattice ideals. Let  $F$  be an  $I$ -fiber. We let

$$I_{<\overline{F}} = \langle B \in I : B \text{ binomial, } \overline{F_B} <_I \overline{F} \rangle$$

and

$$I_{\leq \overline{F}} = \langle B \in I : \overline{F_B} \leq_I \overline{F} \rangle.$$



We note that  $I_{<\overline{F}} = 0$  if and only if there is no  $I$ -fiber  $G$  such that  $\overline{G} <_I \overline{F}$ . It is clear that the definition of these ideals does not depend on the chosen fiber representative. Finally if  $S$  is any subset of binomials of  $I$  we let

$$S_{\overline{F}} = \{B \in S : F_B \in \overline{F}\}.$$

**Remark 3.10.** We will pay extra attention to the fiber that contains 1,  $F_{\{1\}}$ . Let  $S$  be a set of binomials of  $I$ . According to the definitions

$$S_{\overline{F}_{\{1\}}} = \{B \in S : F_B \in \overline{F}_{\{1\}}\} \text{ and } I_{L_{\text{pure}}} = I_{\leq \overline{F}_{\{1\}}}.$$

We isolate the following proposition whose proof is within the proof of [15, Lemma A.1].

**Proposition 3.11.** *Let  $S$  be a minimal generating set of  $I$ ,  $F$  an  $I$ -fiber,  $x^{w_1}, x^{w_2} \in F$ . There exists a subset  $T \subset S_{\overline{F}}$  such that*

$$x^{w_1} - x^{w_2} = \sum_{i,B} \pm x^{a_{i,B}} B$$

where  $B \in T$  may appear more than once,  $a_{i,B} \in \mathbb{N}^n$  and  $a_{i,B} \neq a_{j,B}$  for  $i \neq j$ .

The emphasis of the above statement is that when summing up and factoring out we get an expression

$$x^{w_1} - x^{w_2} = \sum f_i B_i$$

where  $B_i \neq B_j$  for  $i \neq j$  and all nonzero coefficients of the monomial terms of  $f_i$  are  $\pm 1$ .

Next we describe the ideals  $I_{<\overline{F}}$  and  $I_{\leq \overline{F}}$  in terms of the generators of  $I$ .

**Proposition 3.12.** *Let  $S$  be a generating system of binomials for  $I$ . The following hold:*

$$I_{<\overline{F}} = \langle B : B \in S, \overline{F}_B <_I \overline{F} \rangle$$

and

$$I_{\leq \overline{F}} = \langle B : B \in S, \overline{F}_B \leq_I \overline{F} \rangle.$$

*Proof.* We will show the statement for  $I_{<\overline{F}}$ , the other one having a similar proof. Let  $J = \langle B : B \in S, \overline{F}_B <_I \overline{F} \rangle$ . We will show that  $J = I_{<\overline{F}}$ . It is clear that  $J \subset I_{<\overline{F}}$ . To show the other containment it is enough to show that if  $B = x^u - x^v \in I_{<\overline{F}}$  then  $B \in J$ . Let  $B = x^u - x^v \in I_{<\overline{F}}$ . Since  $B \in I$ , by Proposition 3.11 it follows that  $B = \sum_{i=1}^t \pm x^{a_{i,B_i}} B_i$  where  $B_i \in S$  are not necessarily distinct while  $a_{i,B_i} \neq a_{j,B_j}$  for  $i \neq j$ . We will do induction on  $t$ . Without loss of generality we can assume that  $B_1 = x^{u_1} - x^{v_1}$  and  $x^{a_1} x^{u_1} = x^u$ , the other cases being done similarly. First we show the inductive step. Suppose that  $t = 1$ . Since  $x^{a_1} x^{u_1} = x^u$  it follows that  $x^{a_1} F_{B_1} \subset F_B$ . Thus  $\overline{F}_{B_1} \leq_I \overline{F}_B$ . Since  $\overline{F}_B <_I \overline{F}$  we see that  $\overline{F}_{B_1} <_I \overline{F}$ . Assume now that  $t > 1$  and consider  $B' = B - x^{a_1} B_1 = x^{a_1} x^{v_1} - x^v$ . Since  $F_{B'} = F_B$ , it follows that  $B' \in I_{<\overline{F}}$  and we are done by induction.  $\square$

**Theorem 3.13.** *Let  $F$  be an  $I$ -fiber.  $F$  is a Markov  $I$ -fiber if and only if  $I_{<\overline{F}} \neq I_{\leq \overline{F}}$ .*

*Proof.* Let  $S$  be a Markov basis of  $I$ . If  $I_{<\overline{F}} \neq I_{\leq \overline{F}}$  then by Proposition 3.12 there exists a  $B \in S$  such that  $F_B \equiv_I F$ . For the converse assume that there exists  $B \in S$  such that  $\overline{F}_B = \overline{F}$ . It follows immediately that  $I_{<\overline{F}} \neq I_{\leq \overline{F}}$ .  $\square$

**Corollary 3.14.** *The set of equivalence classes of Markov fibers of a lattice ideal  $I$  is an invariant of  $I$ .*

Theorem 4.15 strengthens the above result. For this a more detailed study of the  $I$ -fibers is needed, the scopus of section 4.

#### 4. GENERATING SETS OF LATTICE IDEALS

First we consider the case where the lattice  $L \subset \mathbb{Z}^n$  is generated by its pure elements. We call  $L$  a *pure lattice*. We show how to obtain a generating set of  $I_L$  of least cardinality  $\mu(I_L)$ , i.e. a Markov basis of  $I_L$ . Let  $S = \{B_1, \dots, B_r\}$  be a set of binomials of  $I_L$ . We say that  $S'$  is a *rearrangement* of  $S$  if there is a bijective function  $f : S \rightarrow S'$  such that  $f(B_i) = \pm B_j$ . Compositions of rearrangements is a rearrangement. It is clear that if  $S$  is a generating set of  $I_L$  then all rearrangements of  $S$  are generating sets of  $I_L$ . The theorem below generalizes Lemma 2.1 of [34].

**Theorem 4.1.** *Let  $L = L_{\text{pure}}$  be a pure lattice of rank  $r$ ,  $S$  a set of  $r$  binomials of  $I_L$ ,  $\sigma = \sigma_L$ . The set  $S$  generates  $I_L$  if and only if there is a rearrangement  $\{x^{u_1} - x^{v_1}, x^{u_2} - x^{v_2}, \dots, x^{u_r} - x^{v_r}\}$  of  $S$  such that the following three conditions are satisfied:*

- (1)  $\{u_1 - v_1, \dots, u_r - v_r\}$  is a basis of  $L$ ,
- (2) for  $i \in [r]$ ,  $\text{supp}(u_i) \cup \text{supp}(v_i) \subset \sigma$ ,
- (3)  $x^{v_1} = 1$  and  $\text{supp}(v_i) \subset \bigcup_{j=1}^{i-1} \text{supp}(u_j)$  for  $2 \leq i \leq r$ .

*Proof.* Suppose first that  $S = \{B_1, \dots, B_r\}$  generates  $I_L$ . We let  $B_j = x^{\beta_j} - x^{\gamma_j}$  for  $j \in [r]$ . Let  $u \in L$ . According to Proposition 3.11 there is an index set  $A$  such that

$$x^{u^+} - x^{u^-} = \sum_{l \in A} \pm x^{a_l} (x^{\beta_j} - x^{\gamma_j}),$$

where  $i_l \in [r]$ ,  $a_l \in \mathbb{N}^n$ . Expanding the RHS, equating the exponents of the equal monomial terms, subtracting the expressions for  $u^+$  and  $u^-$  and substituting  $a_l$ ,  $l \in A$ , one gets that  $u \in \mathbb{Z}(\beta_1 - \gamma_1) + \dots + \mathbb{Z}(\beta_r - \gamma_r)$ . This shows that  $L = \mathbb{Z}(\beta_1 - \gamma_1) + \dots + \mathbb{Z}(\beta_r - \gamma_r)$ . Since  $\text{rank}(L) = r$  it follows that  $\{\beta_1 - \gamma_1, \dots, \beta_r - \gamma_r\}$  is a basis of  $L$ . Next we remark that  $\text{supp}(\beta_i) \subset \sigma$  if and only if  $\text{supp}(\gamma_i) \subset \sigma$ . Indeed this is immediate since  $\text{supp}(\beta_i - \gamma_i) \subset \sigma$ . Now suppose that for some  $i \in [r]$ ,  $\text{supp}(\beta_i) \not\subset \sigma$ . We claim that  $B_i = x^{\beta_i} - x^{\gamma_i}$  is redundant in  $S$  as a generator of  $I_L$ . For this we will show that if  $u \in L$  then  $x^{u^+} - x^{u^-}$  can be written as a linear combination of the elements of  $S \setminus B_i$ . Indeed consider again the relation

$$x^{u^+} - x^{u^-} = \sum_l \pm x^{a_l} (B_{j_l})$$

of Proposition 3.11. Substitute the value 0 to any variable  $x_j$  where  $j \notin \sigma$ : the terms in the above relation involving  $B_i$  disappear. Thus  $S \setminus B_i$  is a generating set of  $I_L$ . This is of course a contradiction since the height of  $I_L$  is  $r$ , see [33] by the generalized Krull's Principal Ideal Theorem. To show that there is a rearrangement of  $S$  that satisfies the conditions of the theorem we notice that  $S$  contains a binomial  $B_j$  with 1 as one of its monomial terms. Indeed let  $w \in L^+$  with  $\text{supp}(w) = \sigma$ , see Proposition 2.7. Since  $x^w - 1 \in I_L$ ,  $x^w - 1 = \sum \pm x^{a_l} (B_{i_l})$ . It is clear that there exists a value of  $l$  such that a monomial term of  $B_{i_l}$  is equal to 1 (and  $a_l = 0$ ): otherwise  $x^w - 1 \in \langle x_1, \dots, x_n \rangle$ , a contradiction. It is immediate that we can rearrange  $S$  by a bijective function  $f_1$  so that  $f_1(B_{i_l}) = x^{u_1} - 1$ . Next we claim that there is

$B = x^\beta - x^\gamma \in S$  such that  $B \neq B_{i_l}$  and  $\text{supp}(\beta)$  or  $\text{supp}(\gamma) \subset \text{supp}(u_1)$ . Indeed, suppose not. Then clearly  $\text{supp}(u_1) \neq \sigma$ . Consider again the expression

$$x^w - 1 = \sum_{t \in A, i_t = i_l} x^{a_t} B_{i_l} + \sum_{t \in A, i_t \neq i_l} \pm x^{a_t} (B_{i_t}) .$$

Substitute the value 1 for all variables whose index is in  $\text{supp}(u_1)$  and the value 0 for all other variables. We obtain a contradiction:  $-1 = 0$ . To avoid the contradiction there must be  $B \in S$  so that  $\pm B = x^{u_2} - x^{v_2}$  and  $\text{supp}(v_2) \subset \text{supp}(u_1)$ . We rearrange  $f_1(S)$  by  $f_2$  which keeps all elements of  $f_1(S)$  fixed but  $B$ :  $f_2(B) = x^{u_2} - x^{v_2}$ . More generally once  $f_s$  has been defined for  $s < r$  so that the third condition is satisfied for all  $i \leq s$ , the same argument produces  $f_{s+1}$  with the desired property.

We now prove the converse. Consider a set  $S$  of binomials whose rearrangement  $\{x^{u_1} - 1, x^{u_2} - x^{v_2}, \dots, x^{u_r} - x^{v_r}\}$  satisfies the three conditions of the theorem. Let  $J = \langle x^{u_1} - 1, \dots, x^{u_r} - x^{v_r} \rangle$ . We will show that  $J = I_L$ . It is clear that  $J \subset I_L$ . Since  $u_1, \dots, u_r - v_r$  is a basis of  $L$  and  $\bigcup_{i=1}^r (\text{supp}(u_i) \cup \text{supp}(v_i)) \subset \sigma$  it is clear that  $\bigcup_{i=1}^r (\text{supp}(u_i) \cup \text{supp}(v_i)) = \sigma$ . By the third condition it follows that

$$\bigcup_{i=1}^r \text{supp}(u_i) = \sigma .$$

Next we will show that for every  $k \in [r]$  there exists  $w_k \in L^+$  such that  $x^{w_k} - 1 \in J$  and  $\text{supp}(w_k) = \bigcup_{j=1}^k \text{supp}(u_j)$ . For  $k = 1$  we set  $w_1 = u_1$ . Since  $\text{supp}(v_2) \subset u_1$  there exists  $\lambda_1 \in \mathbb{N}$ ,  $\lambda_1 \gg 0$  such that  $\lambda_1 w_1 > v_2$ . We set  $w_2 = (\lambda_1 w_1 - v_2) + u_2$ ;  $\text{supp}(w_2) = \text{supp}(u_1) \cup \text{supp}(u_2)$ . Moreover

$$x^{w_2} - 1 = x^{\lambda_1 w_1 - v_2} (x^{u_2} - x^{v_2}) + x^{\lambda_1 w_1} - 1 \in J ,$$

as wanted. It is clear that this construction generalizes for all  $k \in [r]$ . In particular  $\text{supp}(w_r) = \sigma$ .

We will now show that if  $u - v \in L$  then  $x^u - x^v \in J$  finishing the proof. Since  $J : (x_1 \cdots x_n)^\infty = I_L$  there exists  $w \in \mathbb{N}^n$  such that  $x^w (x^u - x^v) \in J$ . By the second condition it is clear that  $w$  can be chosen so that  $\text{supp}(w) \subset \sigma$ . Since  $\text{supp}(w_r) = \sigma$  there exists  $\lambda \in \mathbb{N}$ ,  $\lambda \gg 0$  such that  $\lambda w_r > w$ . Therefore  $x^{\lambda w_r} (x^u - x^v) \in J$ . It follows that

$$x^u - x^v = (x^{\lambda w_r} - 1)(x^v - x^u) - x^{\lambda w_r} (x^u - x^v) \in J$$

and consequently  $I_L = J$ . □

We remark that the binomials of a generating set of  $I_L$  when  $L = L_{\text{pure}}$  might have a common monomial factor according to Theorem 4.1. Note also that if  $E = \{u_1, \dots, u_r\}$  is a basis of  $L$  such that  $u_1 \in L^+$  and  $\text{supp}(u_1) = \sigma$  then the set  $\{1 - x^{u_1}, x^{u_2^+} - x^{u_2^-}, \dots, x^{u_r^+} - x^{u_r^-}\}$  is a Markov basis of  $I_L$ . The next corollary states that if  $L = L_{\text{pure}}$ , the ideal  $I_L$  is always a complete intersection, see also [11]. In section 5 we determine when  $I_L$  is a binomial complete intersection ideal for general lattices.

**Corollary 4.2.** *Let  $L = L_{\text{pure}}$  such that  $\text{rank}(L) = r$ . Then  $\mu(I_L) = r$ .*

*Proof.* A basis  $E$  as in the statement of Theorem 4.1 exists by Corollary 2.13. The conclusion follows immediately. □

To characterize the generating sets of lattice ideals we will use the criterion given in [35] and [15]. Let  $L \subset \mathbb{Z}^n$  be a lattice,  $I = I_L$ ,  $\sigma = \sigma_L$ ,  $S$  a subset of  $I$  consisting of vectors of the form  $x^{u^+} - x^{u^-}$  where  $u \in L$ . Let  $F$  be an  $I$ -fiber. The sequence  $(x^{a_1}, x^{a_2}, \dots, x^{a_k})$  is an  $S$ -path from  $x^u$  to  $x^v$  if

- $x^{a_1} = x^u$ ,  $x^{a_k} = x^v$
- for  $j = 1, \dots, k$  each  $x^{a_j}$  in the sequence belongs to the fiber  $F$  and
- $x^{a_j} - x^{a_{j+1}}$  is equal to  $x^{w_j} B_j$  or  $-x^{w_j} B_j$  for some  $B_j \in S$ ,  $w_j \in \mathbb{N}^n$ .

**Theorem 4.3.** ([35] and [15]) *The set  $S$  of binomials of  $I$  is a generating set of  $I$  if and only if for every  $I$ -fiber  $F$  there is an  $S$ -path between any two elements of  $F$ .*

Let  $F$  be an  $I_L$ -fiber and  $G(M_F) = \{x^{a_1}, \dots, x^{a_s}\}$ . We define a relation “ $\sim$ ” among the elements of  $G(M_F)$  as follows:

$$x^{a_i} \sim x^{a_j} \text{ iff } (a_i + L^+) \cap (a_j + L^+) \neq \emptyset.$$

We note that if  $L^+ = \{\mathbf{0}\}$  then  $x^{a_i} \sim x^{a_j}$  only when  $x^{a_i} = x^{a_j}$ .

**Lemma 4.4.** “ $\sim$ ” is an equivalence relation among the elements of  $G(M_F)$ .

*Proof.* It is enough to show transitivity. We can assume that  $L_{\text{pure}} \neq \{\mathbf{0}\}$ , the other case being trivial. Suppose that

$$x^{a_i} \sim x^{a_j} \text{ and } x^{a_j} \sim x^{a_k}.$$

Thus there exist  $u_i, u_j, v_j, v_k \in L^+$  such that

$$a_i + u_i = a_j + u_j \text{ and } a_j + v_j = a_k + v_k.$$

Therefore

$$a_i + (u_i + v_j) = a_k + (u_j + v_k),$$

and we are done.  $\square$

**Lemma 4.5.** Let  $G(M_F) = \{x^{a_1}, \dots, x^{a_s}\}$ . The following holds for the elements of  $G(M_F)$ :

$$x^{a_i} \sim x^{a_j} \text{ if and only if } a_i^\sigma = a_j^\sigma.$$

*Proof.* We can assume that  $L^+ \neq \{\mathbf{0}\}$ , the other case being trivial. Let  $w \in L^+$  such that  $\text{supp}(w) = \sigma$ . Suppose that  $a_i^\sigma = a_j^\sigma$ . Since  $u = a_i - a_j \in L$  it follows that  $u^\sigma = \mathbf{0}$  and thus  $\text{supp}(u) \subset \sigma$ . Therefore there exists  $\lambda \gg 0$  such that  $u + \lambda w \in \mathbb{N}^n$ . Since  $u + \lambda w \in L^+$  and  $a_i + \lambda w = a_j + (u + \lambda w)$  it follows that  $x^{a_i} \sim x^{a_j}$ .

Suppose now that  $x^{a_i} \sim x^{a_j}$ . There exist  $u_i, u_j \in L^+$  such that  $a_i + u_i = a_j + u_j$ . Therefore

$$a_i^\sigma = (a_i + u_i)^\sigma = (a_j + u_j)^\sigma = a_j^\sigma.$$

$\square$

**Lemma 4.6.** Let  $G(M_F) = \{x^{a_1}, \dots, x^{a_s}\}$ . The following holds for the elements of  $G(M_F)$ :

$$x^{a_i} \not\sim x^{a_j} \text{ if and only if } a_i^\sigma, a_j^\sigma \text{ are incomparable.}$$

*Proof.* We will show that  $x^{a_i} \not\sim x^{a_j}$  implies that  $a_i^\sigma$  and  $a_j^\sigma$  are incomparable. Suppose otherwise. Without loss of generality we can assume that  $a_i^\sigma - a_j^\sigma > \mathbf{0}$ . Let  $u = a_i - a_j$ . Let  $w \in L^+$  such that  $\text{supp}(w) = \sigma$ . We can find  $\lambda \in \mathbb{N}$  large enough so that  $(u + \lambda w)_i > 0$  for  $i \in \sigma$ . Since  $\lambda w^\sigma = \mathbf{0}$  and  $u^\sigma > \mathbf{0}$  it follows that  $u + \lambda w > \mathbf{0}$  and thus  $u + \lambda w \in L^+$ . Since  $a_i + \lambda w = a_j + (u + \lambda w)$  and

$\lambda w$ ,  $u + \lambda w \in L^+$  it follows that  $x^{a_i} \sim x^{a_j}$ , a contradiction. The other implication follows immediately from the previous lemma.  $\square$

Lemma 4.5 and 4.6 imply that there are only two possibilities for  $a_i^\sigma, a_j^\sigma$  where  $x^{a_i}, x^{a_j}$  are minimal generators of  $M_F$ : either  $a_i^\sigma = a_j^\sigma$  or  $a_i^\sigma, a_j^\sigma$  are incomparable. If  $X \subset \mathbb{Z}^n$  by  $X^\sigma$  we mean the set whose elements are the vectors  $u^\sigma$  where  $u \in X$ . In particular

$$G(M_F)^\sigma = \{x^{u^\sigma} : x^u \in G(M_F)\}.$$

Note that the cardinality of  $G(M_F)^\sigma$  might be less than the cardinality of  $G(M_F)$ .

**Lemma 4.7.** *Let  $F, F'$  be two equivalent fibers. Then  $G(M_F)^\sigma = G(M_{F'})^\sigma$ .*

*Proof.* Since  $F, F'$  are equivalent  $I$ -fibers, there exist monomials  $x^u, x^v$  such that  $x^u F \subset F'$  and  $x^v F' \subset F$ . Therefore  $x^{u+v} F \subset F$  and  $u + v \in L^+$ . Since  $u, v \in \mathbb{N}^n$  it follows that  $\text{supp}(u), \text{supp}(v) \subset \sigma$ . Let  $G(M_F) = \{x^{a_1}, \dots, x^{a_s}\}$ ,  $G(M_{F'}) = \{x^{b_1}, \dots, x^{b_r}\}$ . To show the desired equality it suffices to show that for any  $i \in [s]$  there is  $j \in [r]$  so that  $a_i^\sigma = b_j^\sigma$ , the other inclusion being taken care by symmetry.

Since  $x^{a_i} x^u$  is in  $F'$  there exists  $j \in [r]$  such that  $x^{b_j}$  divides  $x^{a_i} x^u$ . Therefore  $a_i + u - b_j \in \mathbb{N}^n$ . Since  $\text{supp}(u) \subset \sigma$  it follows that

$$(a_i + u - b_j)^\sigma = a_i^\sigma - b_j^\sigma \geq \mathbf{0}$$

and  $a_i^\sigma \geq b_j^\sigma$ . Similarly there exists  $k \in [s]$  such that  $b_j^\sigma \geq a_k^\sigma$ . Therefore  $a_i^\sigma \geq b_j^\sigma \geq a_k^\sigma$  and  $a_i^\sigma \geq a_k^\sigma$ . It follows that  $a_i^\sigma = a_k^\sigma$  and thus  $a_i^\sigma = b_j^\sigma$ .  $\square$

We also note the following: let  $F$  be an  $I$ -fiber and let  $x^u \in F$ . Even though  $F$  is infinite when  $L_{\text{pure}} \neq \{\mathbf{0}\}$ , we claim that  $u^\sigma$  takes only a finite number of values. The number of values it takes is equal to the cardinality of  $G(M_F)^\sigma$ . Indeed, suppose that  $G(M_F) = \{x^{a_1}, \dots, x^{a_s}\}$ . Since  $x^u \in M_F$  it follows that  $x^u$  is divisible by  $x^{a_i}$  for some  $i \in [s]$ . Thus  $u - a_i \in \mathbb{N}^n$ . Since  $u - a_i \in L$  it follows that  $u - a_i \in L^+$ . Thus  $(u - a_i)^\sigma = \mathbf{0}$ . It follows that  $u^\sigma - a_i^\sigma = \mathbf{0}$  and  $x^{u^\sigma} \in G(M_F)^\sigma$ .

Let  $F$  be any  $I$ -fiber. We construct a graph  $G_{\overline{F}}$  and then we build on  $G_{\overline{F}}$  to construct a graph  $\Gamma_{\overline{F}}$  that will be crucial in determining when a set of binomials of  $I$  generates  $I_{\leq \overline{F}}$ .

**Definition 4.8.** Let  $F$  be an  $I$ -fiber,  $G(M_F)^\sigma = \{x^{a_1^\sigma}, \dots, x^{a_k^\sigma}\}$  where  $x^{a_i} \in G(M_F)$  for  $i \in [k]$ . We define  $G_{\overline{F}} = (V(G), E(G))$  to be the graph with  $V(G) = [k]$ , and

$$E(G) = \{\{i, j\} : \exists x^{u_i}, x^{u_j} \in F \text{ such that } u_i^\sigma = a_i^\sigma, u_j^\sigma = a_j^\sigma, x^{u_i} - x^{u_j} \in I_{< \overline{F}}\}.$$

The graph  $G_{\overline{F}}$  is independent of the fiber representative  $F$ . This is clear for  $V(G)$ , by Lemma 4.7. Next we show the independence for  $E(G)$ . Suppose that  $x^{u_i} - x^{u_j} \in I_{< \overline{F}}$  where  $x^{u_i}, x^{u_j} \in F$  and  $u_i^\sigma = a_i^\sigma, u_j^\sigma = a_j^\sigma$ . Let  $F' \in \overline{F}$  and let  $x^u, x^v$  be such that  $x^u F \subset F'$  and  $x^v F' \subset F$ . By Lemma 3.1,  $\text{supp}(u) \cup \text{supp}(v) \subset \sigma$  and thus  $u^\sigma = v^\sigma = \mathbf{0}$ . Moreover  $x^u x^{u_i} - x^u x^{u_j} \in I_{< \overline{F}}$ ,  $(u + u_i)^\sigma = u^\sigma + u_i^\sigma = u_i^\sigma = a_i^\sigma$ ,  $(u + u_j)^\sigma = a_j^\sigma$  and thus  $E(G)$  is independent on the choice of the fiber representative  $F$ .

**Definition 4.9.** We let  $\Gamma_{\overline{F}}$  to be the complete graph whose vertices are the connected components of  $G_{\overline{F}}$ . Let  $B = x^u - x^v \in I$  such that  $F_B \in \overline{F}$ . We identify  $B$  with an edge of  $\Gamma_{\overline{F}}$  if  $x^{u^\sigma} \neq x^{v^\sigma}$ ,  $B \in I_{\leq \overline{F}}$  and  $B \notin I_{< \overline{F}}$ . We note that different binomials might correspond to the same edge of  $\Gamma_{\overline{F}}$ . For a subset  $S$  of binomials

of  $I$  we denote by  $\Gamma_{\overline{F}}(S)$  the subgraph of  $\Gamma_{\overline{F}}$  induced by the binomials  $B \in S$  such that  $F_B \in \overline{F}$ .

**Lemma 4.10.** *Let  $L \subset \mathbb{Z}^n$  be a lattice,  $I = I_L$ ,  $S$  a binomial subset of  $I$  consisting of binomials so that  $I_{L_{\text{pure}}} = \langle S_{\overline{F}_{\{1\}}} \rangle$  and  $\Gamma_{\overline{F}}(S)$  is a spanning tree of  $\Gamma_{\overline{F}}$  for every  $I$ -fiber  $F$ . Then the set  $S$  is a generating set of  $I$ .*

*Proof.* We will show that for any  $I$ -fiber  $F$  and any  $x^u, x^v \in F$  there is an  $S$ -path between  $x^u, x^v$ . This is clear if  $F \in \overline{F}_{\{1\}}$ . By Theorem 3.8 we can assume that there is an  $S$ -path between any two elements of  $G$  for all  $G$  such that  $\overline{G} <_I \overline{F}$ . We note that  $u - v \in L$ . Suppose that  $G(M_F)^\sigma = \{x^{a_1^\sigma}, \dots, x^{a_k^\sigma}\}$  where  $x^{a_1}, \dots, x^{a_k} \in G(M_F)$ . We examine three cases.

- (1) If  $u^\sigma = v^\sigma$  then  $(u - v)^\sigma = \mathbf{0}$  and by Corollary 2.8 it follows that  $u - v \in L_{\text{pure}}$ . Therefore  $x^u - x^v \in I_{L_{\text{pure}}}$  and since  $I_{L_{\text{pure}}} = \langle S_{\overline{F}_{\{1\}}} \rangle$  it follows that  $x^u - x^v \in \langle S \rangle$ .
- (2) Suppose that  $u^\sigma \neq v^\sigma$  and that the vertices of  $G_{\overline{F}}$  corresponding to  $u^\sigma$  and  $v^\sigma$  are in the same connected component of  $G_{\overline{F}}$ . Assume that  $u^\sigma = a_i^\sigma$  and  $v^\sigma = a_j^\sigma$  and that  $i = i_1, \dots, i_l = j$  is a path in  $G_{\overline{F}}$ . By applying induction on  $l$  it is enough to prove the statement when  $l = 2$  and  $\{i, j\}$  is an edge of  $G_{\overline{F}}$ . It follows from the definition of  $G_{\overline{F}}$  that there exists a binomial  $x^w - x^z \in I_{<\overline{F}}$  such that  $w^\sigma = u^\sigma, z^\sigma = v^\sigma$ . Moreover  $x^w, x^z \in G$  where  $\overline{G} <_I \overline{F}$ . Thus there is a monomial  $x^a$  such that  $x^a G \subset F$ . By case (1) above, there is an  $S$ -path from  $x^u$  to  $x^{w+a}$  and an  $S$ -path from  $x^v$  to  $x^{z+a}$ . By assumption there is an  $S$ -path from  $x^w$  to  $x^z$ , and thus also from  $x^{w+a}$  to  $x^{z+a}$ . Putting the  $S$ -paths together one gets an  $S$ -path from  $x^u$  to  $x^v$ . We point out that the above argument shows that  $x^u - x^v \in I_{<\overline{F}}$ .
- (3) Suppose that  $u^\sigma \neq v^\sigma$  and that the vertices of  $G_{\overline{F}}$  corresponding to  $u^\sigma$  and  $v^\sigma$  are in disconnected components of  $G_{\overline{F}}$ . Since  $S$  determines a spanning tree of  $\Gamma_{\overline{F}}$  there is a series of edges in  $\Gamma_{\overline{F}}$  that leads from the component that corresponds to  $x^{u^\sigma}$  to the component that corresponds to  $x^{v^\sigma}$ . As before it is enough to prove the statement when the components are adjacent. This means that there exists a binomial  $B = x^{u'} - x^{v'} \in (I_{\leq \overline{F}} \setminus I_{<\overline{F}}) \cap S$  such that  $u'^\sigma, u^\sigma$  correspond to the same connected component of  $G_{\overline{F}}$  and similarly for  $v'^\sigma, v^\sigma$ . The monomials  $x^{u'}, x^{v'}$  of  $B$ , belong to a fiber equivalent to  $F$ . It follows that there is  $b \in \mathbb{N}^n$  such that  $x^{u'+b}$  and  $x^{v'+b}$  belong to  $F$  and thus the sequence  $(x^{u'+b}, x^{v'+b})$  is an  $S$ -path from  $x^{u'+b}$  to  $x^{v'+b}$ . By case (2) above, there is an  $S$ -path from  $x^u$  to  $x^{u'+b}$  and an  $S$ -path from  $x^v$  to  $x^{v'+b}$ . Joining these paths one gets an  $S$ -path from  $x^u$  to  $x^v$ .

□

We let  $t(\overline{F})$  denote the number of vertices of  $\Gamma_{\overline{F}}$ . Thus

$$t(\overline{F}) := |V(\Gamma_{\overline{F}})|.$$

We note that to construct a spanning tree of  $\Gamma_{\overline{F}}$  we need exactly  $t(\overline{F}) - 1$  binomials. To prove the next theorem we will use Theorem 4.3.

**Theorem 4.11.** *Let  $L \subset \mathbb{Z}^n$  be a lattice,  $I = I_L$ ,  $S$  a subset of  $I$  consisting of binomials. The set  $S$  is a Markov basis of  $I$  if and only if the following conditions are satisfied:*

- $\Gamma_{\overline{F}}(S)$  is a spanning tree of  $\Gamma_{\overline{F}}$  for every  $I$ -fiber  $F$  and  $|S_{\overline{F}}| = t(\overline{F}) - 1$ ,
- $|S_{\overline{F}_{\{1\}}}| = \text{rank}(L_{\text{pure}})$  and
- $I_{L_{\text{pure}}} = \langle S_{\overline{F}_{\{1\}}} \rangle$ .

*Proof.* Suppose that  $S$  is a Markov basis of  $I$ . We will show that  $S$  satisfies the three conditions of the theorem. We note that since  $S$  is a generating set of  $I$ , by Proposition 3.12 and Remark 3.10 it follows that  $\langle S_{\overline{F}_{\{1\}}} \rangle = I_{L_{\text{pure}}}$ . Moreover if  $|S_{\overline{F}_{\{1\}}}| > \text{rank}(L_{\text{pure}})$ , then by Theorem 4.1 one can replace the binomials in  $S_{\overline{F}_{\{1\}}}$  by a Markov basis of  $I_{L_{\text{pure}}}$ . By Lemma 4.10 the new set thus produced is still a generating set of  $I$  and has smaller cardinality than  $S$ , a contradiction. Next we show that for an arbitrary  $I$ -fiber  $F$ ,  $\Gamma_{\overline{F}}(S)$  is a spanning tree of  $\Gamma_{\overline{F}}$ . Indeed, by Theorem 4.3,  $S$  induces a spanning graph in  $F$ . Since  $G_{\overline{F}}$  comes from  $F$  by identifying components and similarly for  $\Gamma_{\overline{F}}$  from  $G_{\overline{F}}$ , it follows that  $\Gamma_{\overline{F}}(S)$  is a spanning graph of  $\Gamma_{\overline{F}}$ . We show that  $\Gamma_{\overline{F}}(S)$  is a tree of  $\Gamma_{\overline{F}}$ . If not  $\Gamma_{\overline{F}}(S)$  has a cycle in  $\Gamma_{\overline{F}}$ . We omit from  $S$  the binomial that induces an edge on this cycle. The resulting set still satisfies the conditions of Lemma 4.10 and is thus a generating set of  $I$  of smaller cardinality, a contradiction. Similarly if  $|S_{\overline{F}}| > t(\overline{F}) - 1$  there is a binomial in  $S_{\overline{F}}$  that does not correspond to an edge of  $\Gamma_{\overline{F}}(S)$  or two binomials that correspond to the same edge. Then one binomial could be omitted from  $S$  and the resulting set would still be a generating set of  $I$ .

Conversely let  $S$  be a set that satisfies the three conditions of the theorem. By Lemma 4.10,  $S$  is a generating set of  $I$ . Suppose that there is a Markov basis  $S'$  of  $I$  such that  $|S'| < |S|$ . Let  $F$  be a Markov fiber such that  $|S'_{\overline{F}}| < |S_{\overline{F}}|$ . We note that  $F \notin \overline{F}_{\{1\}}$  since  $|S'_{\overline{F}_{\{1\}}}| \geq \text{rank}(L_{\text{pure}}) = |S_{\overline{F}_{\{1\}}}|$ . Moreover  $|S_{\overline{F}}| = t(\overline{F}) - 1 = |S'_{\overline{F}}|$ , thus  $|S_{\overline{F}}| = |S'_{\overline{F}}|$ , a contradiction.  $\square$

**Remark 4.12.** When  $L_{\text{pure}} = \{0\}$  then the construction and conditions on  $\Gamma_{\overline{F}}$  coincide with the construction and conditions of [4] and [8] since by Proposition 2.3  $G(M_F) = F$ .

We remark that for all but finitely many equivalence classes of fibers  $\overline{F}$ ,  $t(\overline{F}) = 1$ . Indeed by Corollary 3.14 the set consisting of equivalence classes of Markov  $I$ -fibers is finite. If an  $I$ -fiber  $F$  is not a Markov fiber, then by Theorem 3.13 it follows that  $I_{\leq \overline{F}} = I_{< \overline{F}}$  and hence  $G_{\overline{F}}$  consists of only one connected component. The next result is the main theorem of this section. Its proof is an immediate consequence of Theorem 4.11.

**Theorem 4.13.** *Let  $L \subset \mathbb{Z}^n$  be a lattice. Then*

$$\mu(I_L) = \text{rank}(L_{\text{pure}}) + \sum_{\overline{F} \neq \overline{F}_{\{1\}}} (t(\overline{F}) - 1),$$

*where the sum runs over all distinct equivalence classes of Markov fibers.*

The next corollary follows from Corollary 3.14 and Theorem 4.11. It generalizes the corresponding result for positively graded lattices, see [10].

**Corollary 4.14.** *Let  $L \subset \mathbb{Z}^n$  be a lattice,  $\mu = \mu(I_L)$ ,  $\{B_1, \dots, B_\mu\}$  a Markov basis of  $I_L$ . The multiset*

$$\{\overline{F}_{B_1}, \dots, \overline{F}_{B_\mu}\}$$

*is an invariant of  $I_L$ .*

The following result concerns an arbitrary minimal generating set of  $I_L$ : the play in the cardinality of such a set only concerns the pure part of  $L$ . The proof of Corollary 4.15 follows directly from the proof of Theorem 4.11.

**Corollary 4.15.** *Let  $L$  be a lattice and  $S = \{B_1, \dots, B_t\}$  a minimal generating set of  $I_L$ . The multiset*

$$\{\overline{F}_{B_i} : F_{B_i} \notin \overline{F}_{\{1\}}\}$$

*is an invariant of  $I_L$ .*

Next we examine the binomials that appear in every Markov basis of  $I$ .

**Definition 4.16.** A binomial is called *indispensable* if it appears in every Markov basis of  $I$  up to a constant multiple. A monomial  $x^u$  is called *indispensable* if for every Markov basis  $S$  of  $I$  there is a binomial  $B \in S$  so that  $x^u$  is a monomial term of  $B$ .

If  $\text{rank}(L_{\text{pure}}) = 0$ , the characterization of indispensable binomials is given by [4, Corollary 2.10], see also [4, Theorem 3.4].

**Theorem 4.17.** *Let  $L$  be a lattice. If  $\text{rank}(L_{\text{pure}}) > 1$  then there are no indispensable binomials and only one indispensable monomial,  $x^0$ . If  $\text{rank}(L_{\text{pure}}) = 1$  there exists exactly one indispensable binomial and exactly two indispensable monomials.*

*Proof.* Let  $L$  be a lattice such that  $\text{rank}(L_{\text{pure}}) = r \geq 1$ ,  $S$  a Markov basis of  $I$ . If  $L_{\text{pure}}$  has rank 1 then  $L_{\text{pure}} = \langle u \rangle$ , where  $u \in L^+$  is  $L$ -primitive and by Theorem 4.1 we have that  $S_{\overline{F}_1} = \{x^u - 1\}$ .

Suppose that  $L_{\text{pure}}$  has rank  $r > 1$ . Without loss of generality we can assume that  $S_{\overline{F}_1} = \{x^{u_1} - 1, \dots, x^{u_r} - x^{v_r}\}$  satisfies directly the three conditions of Theorem 4.1. For every  $i \geq 2$  let  $u'_i = u_i + u_1$  and  $v'_i = v_i + u_1$  and note that  $u'_i - v'_i = u_i - v_i$ . By Theorem 4.1 it follows that  $\{x^{u_1} - 1, x^{u'_2} - x^{v'_2}, \dots, x^{u'_r} - x^{v'_r}\}$  is also a Markov basis of  $I_{L_{\text{pure}}}$ . Since  $\text{rank}(L_{\text{pure}}) > 1$  there are infinitely many  $L$ -primitive elements of full support and thus infinitely many bases of  $L_{\text{pure}}$  as in Corollary 2.13. By applying the above argument to a rearrangement of any of these bases we conclude that there are no indispensable binomials for  $I_{L_{\text{pure}}}$ . The only indispensable monomial of  $I_{L_{\text{pure}}}$  is  $1 = x^0$ .

Finally we show that if there is a Markov fiber  $F$  such that  $\overline{F} >_I \overline{F}_{\{1\}}$  then there are infinitely many distinct choices for the binomials that determine any edge in a spanning tree of  $\Gamma_{\overline{F}}$ . Since  $F$  is a Markov fiber, Theorem 3.13 says that  $I_{<\overline{F}} \neq I_{<\overline{F}_1}$ . Thus there exists  $B = x^u - x^v$ , such that  $F_B \in \overline{F}$  and  $B \notin I_{<\overline{F}}$ . We note that  $u^\sigma \neq v^\sigma$ : otherwise  $u - v \in L_{\text{pure}}$  and  $x^u - x^v \in I_{L_{\text{pure}}} \subset I_{<\overline{F}}$ , a contradiction. Thus  $B$  produces an edge in  $\Gamma_{\overline{F}}$  and can be made part of a Markov basis  $S$  of  $I$ . Let  $w_1, w_2 \in L^+$ . Then  $B' = x^{u+w_1} - x^{v+w_2}$  gives exactly the same edge as  $B$  and can replace  $B$  in  $S$ . Therefore, applying Theorem 4.11 we obtain that there are no indispensable binomials of  $I$ , but there is exactly one indispensable monomial of  $I$ , that is  $1 = x^0$ .  $\square$

We isolate the following result which follows from the proof of Theorem 4.17.

**Theorem 4.18.** *Let  $L$  be a lattice. If  $\text{rank}(L_{\text{pure}}) > 1$  or  $\text{rank}(L_{\text{pure}}) = 1$  and  $L \neq L_{\text{pure}}$  then the Universal Markov basis of  $I_L$  is infinite.*

We note that for the lattices  $L$  from Theorem 4.18 the Universal Markov basis of  $I_L$  is not contained in the Graver basis, since the Graver basis is finite (see [22]).



## 5. BINOMIAL COMPLETE INTERSECTION LATTICE IDEALS

In this section we determine binomial complete intersection lattice ideals. This is a problem that engaged mathematicians starting in 1970, see [14]. We recall that  $\sigma = \sigma_L$  is the maximum support of an element of  $L \cap \mathbb{N}^n$  and that  $L_{pure}$  is the sublattice of  $L$  generated by the elements of  $L \cap \mathbb{N}^n$ . By  $L^\sigma$  we mean the sublattice of  $(\mathbb{N}^n)^\sigma$  generated by the vectors  $u^\sigma$  where  $u \in L$ . We first show that the lattice ideal of  $L^\sigma$  is positively graded.

**Remark 5.1.**  $L^\sigma \cap (\mathbb{N}^n)^\sigma = \{\mathbf{0}\}$ .

*Proof.* Let  $w \in L \cap \mathbb{N}^n$  such that  $\text{supp}(w) = \sigma$ . If  $\mathbf{0} \neq u^\sigma \in L^\sigma \cap (\mathbb{N}^n)^\sigma$  then there exists  $k \in \mathbb{N}$ ,  $k \gg 0$  such that  $u' = u + kw \in L \cap \mathbb{N}^n$ . Thus  $\sigma \subsetneq \text{supp}(u')$ , a contradiction.  $\square$

Next we show that the rank of  $L$  is determined by the ranks of its sublattice  $L_{pure}$  and the lattice  $L^\sigma$ .

**Proposition 5.2.** *Let  $L \subset \mathbb{Z}^n$  be a lattice. Then*

$$\text{rank}(L) = \text{rank}(L^\sigma) + \text{rank}(L_{pure}).$$

*Proof.* Let  $\mathbb{B}_1 = \{u_1, \dots, u_k\} \subset L$  be such that  $\mathbb{B}'_1 = \{u_1^\sigma, \dots, u_k^\sigma\}$  is a basis of  $L^\sigma$  and let  $\mathbb{B}_2 = \{v_1, \dots, v_r\}$  be a basis of  $L_{pure}$ . We will prove our statement by showing that  $\mathbb{B} = \mathbb{B}_1 \cup \mathbb{B}_2$  is a basis of  $L$ . In order to prove this we first show that  $\mathbb{B}$  is linearly independent. Indeed, let  $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_r$  be integers such that

$$\alpha_1 u_1 + \dots + \alpha_k u_k + \beta_1 v_1 + \dots + \beta_r v_r = \mathbf{0}.$$

Then

$$\alpha_1 u_1^\sigma + \dots + \alpha_k u_k^\sigma + \beta_1 v_1^\sigma + \dots + \beta_r v_r^\sigma = \mathbf{0},$$

and since  $v_i^\sigma = \mathbf{0}$  for  $i \in [r]$  it follows that

$$\alpha_1 u_1^\sigma + \dots + \alpha_k u_k^\sigma = \mathbf{0},$$

Since  $\mathbb{B}'_1$  is a basis of  $L^\sigma$  it follows that  $\alpha_i = 0$  for  $i \in [k]$ . Hence  $\beta_1 v_1 + \dots + \beta_r v_r = \mathbf{0}$  and thus  $\beta_j = 0$  for  $j \in [r]$  since  $\mathbb{B}_2$  is a basis of  $L_{pure}$ .

It remains to show that  $\mathbb{B}$  is a system of generators for  $L$ . Let  $v \in L$  be an arbitrary vector. Then  $v^\sigma = \lambda_1 u_1^\sigma + \dots + \lambda_k u_k^\sigma$ , where  $\lambda_i \in \mathbb{Z}$  for  $i \in [k]$ . Consider

$$u = v - \lambda_1 u_1 - \dots - \lambda_k u_k.$$

Since  $u^\sigma = \mathbf{0}$  it follows that  $\text{supp}(u) \subset \sigma$ . By Proposition 2.8 it follows that  $u \in L_{pure}$ . Consequently  $v \in \langle \mathbb{B} \rangle$ .  $\square$

We consider the lattice ideal  $I_{L^\sigma}$  in  $R^\sigma := \mathbb{k}[x_i : i \notin \sigma]$ . We show that in order to compute the  $I_{L^\sigma}$ -fibers, it is enough to consider the generating sets of the corresponding  $I_L$ -fibers.

**Lemma 5.3.** *Let  $u \in \mathbb{N}^n$ . Then the  $I_{L^\sigma}$ -fiber of  $u^\sigma$  is  $G(M_{F_u})^\sigma$ .*

*Proof.* Let  $u' = u^\sigma \in (\mathbb{N}^n)^\sigma$  and denote by  $F'$  the  $I_{L^\sigma}$ -fiber of  $u'$ . It follows from Remark 5.1 that  $F'$  is finite. We will show that  $F' = F_u^\sigma$  and thus by Proposition 2.6 we obtain  $F' = G(M_{F_u})^\sigma$ , since for any vector  $t \in L^+$  we have  $t^\sigma = \mathbf{0}$ . To prove this, consider first an element  $v \in F_u$ . Then  $v - u \in L$  and  $v^\sigma - u^\sigma \in L^\sigma$ . Hence  $v^\sigma \in F'$  and we obtain the inclusion  $F' \supset F_u^\sigma$ . For the converse inclusion, let  $v' \in F'$ . Then  $u' - v' \in L^\sigma$  and it follows that there exists a vector  $w \in L$  such

that  $u' - v' = w^\sigma$ . Therefore  $v' = (u - w)^\sigma$ . Since  $u - (u - w) = w \in L$  we obtain that  $u - w \in F_u$  and  $v' \in F_u^\sigma$ , as desired.  $\square$

Next we show that the cardinality of a Markov basis of  $I_L$  depends on the cardinality of a Markov basis of  $I_{L^\sigma}$ .

**Theorem 5.4.** *Let  $L$  be a lattice. Then*

$$\mu(I_L) = \mu(I_{L^\sigma}) + \text{rank}(L_{\text{pure}}) .$$

*Proof.* If  $\overline{F} \neq \overline{F}_{\{1\}}$  then by Lemma 5.3 the graphs  $\Gamma_{\overline{F}_u}$  and  $\Gamma_{\overline{F}_{u^\sigma}}$  are equal. The theorem follows from Theorem 4.13.  $\square$

The following theorem follows directly from Proposition 5.2 and Theorem 5.4 and determines the binomial complete intersection lattice ideals. It is the main theorem of this section.

**Theorem 5.5.** *Let  $L \subset \mathbb{Z}^n$  be a lattice. The ideal  $I_L$  is binomial complete intersection if and only if  $I_{L^\sigma}$  is complete intersection.*

We can describe the lattices for which  $I_L$  is a binomial complete intersection. Recall that a mixed dominating matrix  $M$  has the property that every row of  $M$  has a positive and negative entry and  $M$  contains no square submatrix with this property.

**Corollary 5.6.** *Let  $L \subset \mathbb{Z}^n$  be a lattice. The ideal  $I_L$  is binomial complete intersection if and only if there exists a basis of  $L^\sigma$  so that its vectors give the rows of a mixed dominating matrix.*

*Proof.* By Remark 5.1,  $I_{L^\sigma}$  is positively graded. The proof now follows from Theorem 5.5 and [23, Theorem 3.9].  $\square$

**Remark 5.7.** Let  $L \subset \mathbb{Z}^n$  be a lattice,  $r = \text{rank}(L)$ ,  $r_+ = \text{rank}(L_{\text{pure}})$ ,  $\sigma = \sigma_L$ . The previous corollary states that  $I_L$  is binomial complete intersection if and only if there is a basis of  $L$  whose vectors give the rows of

$$\begin{bmatrix} A & M \\ C & \mathbf{0} \end{bmatrix}$$

where  $A \in \mathcal{M}_{(r-r_+) \times |\sigma|}(\mathbb{Z})$ , the matrix  $M \in \mathcal{M}_{(r-r_+) \times (n-|\sigma|)}(\mathbb{Z})$  is mixed dominating, the matrix  $C \in \mathcal{M}_{r_+ \times |\sigma|}(\mathbb{Z})$  is a matrix whose rows satisfy the conditions of Theorem 4.1, and  $\mathbf{0}$  is the zero matrix. For example, any matrix with entries in  $\mathbb{N}$  and independent rows has the desired property for  $C$  above, while for a mixed dominating matrix  $M$  one can use [23, Remark 3.17] or [12, Theorem 2.2]. By working with the appropriate size matrices one can easily obtain a class of lattice ideals that are binomial complete intersections.

## 6. EXAMPLE

Let  $L$  be the sublattice of  $\mathbb{Z}^5$  generated by the vectors  $v_1 = (3, 0, 1, -1, 0)$ ,  $v_2 = (0, 1, 6, 0, -1)$ ,  $v_3 = (1, 1, 0, 0, 0)$  and  $v_4 = (5, 0, 0, 0, 0)$ . Let  $v_5 = (0, 5, 0, 0, 0)$ . It is not hard to show that

$$L^+ = L \cap \mathbb{N}_0^5 = \mathbb{N}_0 v_3 + \mathbb{N}_0 v_4 + \mathbb{N}_0 v_5$$

while

$$L_{\text{pure}} = \mathbb{Z} v_3 + \mathbb{Z} v_4 .$$

Thus  $\text{rank}(L_{\text{pure}}) = 2$ ,  $\sigma := \sigma_L = \{1, 2\}$  and according to Theorem 4.1

$$I_{L_{\text{pure}}} = \langle 1 - x_1^5, 1 - x_1 x_2 \rangle.$$

To find a generating set of  $I_L$  in  $R = \mathbb{k}[x_1, \dots, x_5]$  we compute the ideal

$$\langle x_4 - x_1^3 x_3, x_5 - x_2 x_3^6, 1 - x_1 x_2, 1 - x_1^5 \rangle : (x_1 \cdots x_5)^\infty$$

using CoCoA [6]. It turns out that  $I_L$  is generated by  $\{B_1, \dots, B_{16}\}$  where

$$\begin{aligned} B_1 &= x_2^3 - x_1^2, & B_2 &= x_1^3 - x_2^2, & B_{14} &= x_1 x_2 - 1, \\ B_3 &= x_1^2 x_4 - x_3, & B_6 &= x_2 x_3 - x_1 x_4, & B_{15} &= x_2^2 x_4 - x_1 x_3, & B_{16} &= x_1^2 x_3 - x_2 x_4 \\ B_4 &= x_2 x_4^2 - x_3^2, & B_5 &= x_1 x_3^2 - x_4^2, & B_7 &= x_1 x_4^3 - x_3^3, & B_8 &= x_3^5 - x_4^5, \\ B_9 &= x_3^2 x_4^4 - x_2^2 x_5, & B_{10} &= x_3^3 x_4^3 - x_5, & B_{11} &= x_3^4 x_4^2 - x_1^2 x_5, & B_{12} &= x_4^6 - x_2 x_5, \\ B_{13} &= x_3 x_4^5 - x_1 x_5 \end{aligned}$$

(We wrote the binomials in the order they appear in CoCoA).

Let  $F$  be an  $I_L$ -fiber. Since  $|\mathbb{Z}^2/(L_{\text{pure}})_\sigma| = 5$  it follows by Proposition 3.5 that  $\overline{F}$  consists of 5 equivalent fibers. In particular let  $F_{\{1\}}$  be the fiber that contains the identity. As in Example 3.6(b) we see that

$$\overline{F}_{\{1\}} = \{F_{\{1\}}, F_{x_1}, F_{x_1^2}, F_{x_1^3}, F_{x_1^4}\}$$

where for each  $0 \leq k \leq 4$ , the fiber  $F_{x_1^k}$  consists of the monomials  $x_1^i x_2^j$  with  $i - j \equiv k \pmod{5}$ . Thus if  $x^u \in R$  then

$$\overline{F}_{x^u} = \{F_{x^u}, F_{x^u x_1}, F_{x^u x_1^2}, F_{x^u x_1^3}, F_{x^u x_1^4}\}.$$

Of the generators  $B_i$  of  $I_L$  we notice that

- $F_{B_1}, F_{B_2}, F_{B_{14}} \in \overline{F}_{\{1\}}$
- $F_{B_3}, F_{B_6}, F_{B_{15}}, F_{B_{16}} \in \overline{F}_{x_4}$ ,
- $F_{B_4}, F_{B_5} \in \overline{F}_{x_4^2}$ ,
- $F_{B_7} \in \overline{F}_{x_4^3}$ ,
- $F_{B_8} \in \overline{F}_{x_4^5}$  and
- $F_{B_9}, \dots, F_{B_{13}} \in \overline{F}_{x_4^6}$ .

It is clear that the above equivalence classes of fibers are pairwise distinct: use Lemma 3.1 and notice that there is no  $w \in \mathbb{N}^5$  with  $\text{supp}(w) \subset \sigma$  such that  $w + (0, 0, 0, k, 0) \in L$ . It is easy to see that

$$\overline{F}_{\{1\}} <_I \overline{F}_{x_4} <_I \overline{F}_{x_4^2} <_I \overline{F}_{x_4^3} <_I \overline{F}_{x_4^5} <_I \overline{F}_{x_4^6}.$$

We also note that  $\overline{F}_{x_4^6} = \overline{F}_{x_5}$ . For  $i \in [6]$  we compute  $I_{<\overline{F}_{x_4^i}}$  and  $I_{\leq \overline{F}_{x_4^i}}$  with the use of CoCoA and apply Theorem 3.13 to conclude that  $F_{\{1\}}$ ,  $F_{x_4}$  and  $F_{x_5}$  are Markov fibers. Next we show how to obtain a Markov basis  $S$  of  $I_L$  using the generating set  $\{B_1, \dots, B_{16}\}$ .

According to Theorem 4.11,  $|S_{\overline{F}_{\{1\}}}| = \text{rank}(L_{\text{pure}}) = 2$  and  $S_{\overline{F}_{\{1\}}}$  must generate  $I_{L_{\text{pure}}} = \langle 1 - x_1^5, 1 - x_1 x_2 \rangle$ . We note that there are infinitely many choices for binomials  $B, B'$  that generate  $I_{L_{\text{pure}}}$ , see Theorem 2.12, Theorem 4.1 and Proposition 2.10. We remark that the multiset  $\{F_B, F_{B'}\}$  equals  $\{F_{\{1\}}, F\}$  where  $F$  can be any of the five fibers of  $\overline{F}_{\{1\}}$ .

Consider now the fiber  $F_{x_4}$ . It is an easy exercise that

$$G(M_{F_{x_4}}) = \{x_2^2 x_3, x_1^3 x_3, x_4\}.$$

Thus  $G(M_{F_{x_4}})^\sigma = \{x_4, x_3\}$ . Since  $I_{<\overline{F}_{x_4}} = I_{L_{pure}}$  it is immediate that  $G_{\overline{F}_{x_4}}$  consists of 2 isolated vertices. Thus  $t(\overline{F}_{x_4}) = 2$  and exactly one binomial is needed to construct a spanning tree of  $\Gamma_{\overline{F}_{x_4}}$ . To obtain a Markov basis  $S$  of  $I_L$ , according to Theorem 4.11 we need to add to  $S_{\overline{F}_{\{1\}}}$  a binomial  $\pm(x^u - x^v)$  such that  $x^u \in x_3 F_{x_1^{3+i}}$  and  $x^v \in x_4 F_{x_1^i}$ , where  $0 \leq i \leq 4$ . For example any of  $B_3, B_6, B_{15}, B_{16}$  are of the required type. Let  $S' = S_{\overline{F}_{\{1\}}} \cup \{x^u - x^v\}$  be this set.

Next consider the fiber  $F_{x_5}$ . It can be shown that  $G(M_{F_{x_5}})$  is the set  $\{x_5, x_1^4 x_3^6, x_1 x_3^5 x_4, x_1^3 x_3^4 x_4^2, x_3^3 x_4^3, x_1^2 x_3^2 x_4^4, x_1^4 x_3 x_4^5, x_1 x_4^6, x_2 x_3^6, x_2^4 x_3^5 x_4, x_2^2 x_3^4 x_4^2, x_2^3 x_3^2 x_4^4, x_2 x_3 x_4^5, x_2^4 x_4^6\}$  and thus

$$G(M_F)^\sigma = \{x_5, x_3^6, x_3^5 x_4, x_3^4 x_4^2, x_3^3 x_4^3, x_3^2 x_4^4, x_3 x_4^5, x_4^6\}.$$

We claim that the graph  $G_{\overline{F}_{x_5}}$  consists of two connected components: the isolated vertex  $x_5$  and a component containing all other vertices. We show for example that there is an edge between  $x_3^6$  and  $x_3^5 x_4$ :

$$x_1^4 x_3^6 - x_1 x_3^5 x_4 = x_1 x_3^5 (x_1^3 x_3 - x_4), \quad x_1^3 x_3 - x_4 \in I_{\leq \overline{F}_{x_4}} \quad \text{and} \quad \overline{F}_{x_4} <_I \overline{F}_{x_5}.$$

We note that  $x_5$  is necessarily an isolated vertex, otherwise  $G_{\overline{F}_{x_5}}$  would be connected, a contradiction by Theorem 4.11 since  $F_{x_5}$  is a Markov fiber. It is easy to see that any of  $B_9, \dots, B_{13}$  would produce an edge among the two connected components of  $G_{\overline{F}_{x_5}}$ . To obtain a Markov basis  $S$  of  $I_L$ , according to Theorem 4.11 we need to add to  $S'$  exactly one binomial  $\pm(x^{w_1} - x^{w_2})$  such that  $x^{w_1} \in x_5 F_{x_1^i}$  and  $x^{w_2}$  belongs in the union of the sets  $x_3^6 F_{x_1^{4+i}}, x_3^5 x_4 F_{x_1^{1+i}}, x_3^4 x_4^2 F_{x_1^{3+i}}, x_3^3 x_4^3 F_{x_1^i}, x_3^2 x_4^4 F_{x_1^{2+i}}, x_3 x_4^5 F_{x_1^{1+i}}, x_4^6 F_{x_1^{1+i}}$ , where  $0 \leq i \leq 4$ . For example any of  $B_9, \dots, B_{13}$  are of the required type.

The cardinality of  $S$  is 4 and  $I_L$  is a binomial complete intersection. Indeed, this follows immediately from Corollary 5.6 since  $L^\sigma$  is generated by  $v_1^\sigma, v_2^\sigma$  and the matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ 6 & 0 & -1 \end{bmatrix}$$

is mixed dominating.

We presented the example in great detail, since the different steps illuminate the various parts of the proof of Theorem 4.11.

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